Uniaxial nematic elastomers: constitutive framework and a simple application

BY Yi-chao Chen1 AND Eliot Fried2,*

1Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4006, USA
2Department of Mechanical and Aerospace Engineering, Washington University in St. Louis, St. Louis, MO 63130-4899, USA

Using a geometrical approach to treat the constraints associated with the continuum description of a uniaxial nematic elastomer—that is, a nematic elastomer whose microstructure can be characterized by a single unit vector field, we develop the general constitutive framework for such a material. We also provide a variational formulation of equilibrium and use that formulation to discuss the inflation of a tubular specimen. The analysis demonstrates the existence of a pair of disconnected solution branches. Of these branches, one is stable at low pressure and the other is stable at high pressure. An intermediate regime within which both branches are stable indicates that during loading the specimen may undergo a discontinuous jump from one solution branch to another and thus suggests the possibility of using nematic elastomers for pressure-sensitive optical switches.

Keywords: nematic elastomers; constitutive equations; material constraints; stability; optical switching

1. Introduction

Nematic elastomers combine the properties of nematic liquid crystals and elastomers to yield anisotropic networks. The polymer chains constituting such networks are formed by connecting mesogens end-to-end or by hanging mesogens from pre-existing chains. As in a conventional elastomer, the chains of a nematic elastomer are cross-linked. A comprehensive introduction to nematic elastomers is provided by Warner & Terentjev (2003).

From a continuum-mechanical perspective, a nematic elastomer can be modelled as having (i) macroscopic degrees of freedom associated with network distortion and (ii) additional microscopic degrees of freedom associated with nematic ordering. The kinematical state of the nematic elastomer is thus described by two fields:

(i) the deformation, \( y \), from the region occupied by the body in a given reference configuration to the region occupied by the body in the spatial configuration; and

* Author for correspondence (efried@me.wustl.edu).
(ii) the conformation, $A$, a symmetric and positive-definite tensorial measure of the average shape of the polymer molecules in the spatial configuration.

To account for the possibility of the nematically induced anisotropy of the reference configuration, a referential conformation $A_s$ is also introduced. In contrast to $A$, which is a kinematical field that describes nematically induced distortion and anisotropy in the spatial configuration, $A_s$ embodies symmetry properties of the material.

We confine our attention to nematic elastomers that are uniaxial in the reference and spatial configurations, in which case the conformation tensors admit the representations

$$A_s = a_s(1 + q_s)^{-1/3}(I + q_s n_s \otimes n_s), \quad a_s > 0, \quad |n_s| = 1, \quad q_s > -1, \quad (1.1)$$

and

$$A = a(1 + q)^{-1/3}(I + q n \otimes n), \quad a > 0, \quad |n| = 1, \quad q > -1. \quad (1.2)$$

In (1.2), $n$ and $q$ represent the average molecular orientation and asphericity in the spatial configuration, and $a=(\det A)^{1/3}$. Geometrically, $A$ as defined by (1.2) corresponds to an ellipsoid of revolution on the axis determined by $n$. That ellipsoid is oblate for $-1 < q < 0$, spherical for $q=0$, and prolate for $q>0$. Remarks analogous to those made concerning $a$, $n$, and $q$ apply to $a_s$, $n_s$, and $q_s$. A continuum theory for uniaxial nematic elastomers has been developed by Anderson et al. (1999). In that theory, the representation (1.2) for the conformation tensor is used with the orientation vector $n$ taken as a basic kinematic variable. Equation (1.2)$_3$ is then treated as a constraint on the orientation $n$. On writing $G=\grad n$ for the orientation gradient, the auxiliary constraint,

$$G^T n = 0, \quad (1.3)$$

follows from (1.2)$_3$. Moreover, the material is assumed to be incompressible, so that the deformation gradient $F=\grad y$ obeys

$$\det F = 1. \quad (1.4)$$

These considerations lead Anderson et al. (1999) to define a manifold via three scalar constraint equations. A three-dimensional linear space $T$ is then defined through the derivatives of the constraint equations. The space of triples $(F, n, G)$ is subsequently written as the direct sum of $T$ and its orthogonal complement $T^\perp$, the latter being an 18-dimensional linear space. Furthermore, the triple $(S, -\pi, \Sigma)$ determined by the deformational stress $S$, internal orientational body force density $\pi$, and orientational stress $\Sigma$ is decomposed additively into the sum of an active component $(S_a, -\pi_a, \Sigma_a)$ and a reactive component $(S_r, -\pi_r, \Sigma_r)$. Of fundamental importance in this work is the notion that $(S_r, -\pi_r, \Sigma_r)$ is powerless during any process consistent with the constraints. The reactive and active components of $(S, -\pi, \Sigma)$ belong to $T$ and $T^\perp$, respectively. Moreover, only $(S_a, -\pi_a, \Sigma_a)$ is constitutively determinate.

While the basic ideas employed in Anderson et al. (1999) are conceptually sound, the derivation of the constitutive functions is incomplete in one aspect: because the auxiliary constraint (1.3) is replaced by a scalar equation $u \cdot G^T n = 0$, with $u$ being an arbitrary vector, the linear space $T$ and the orthogonal projection onto $T^\perp$ both depend on $u$. Projecting the triple $(S, -\pi, \Sigma)$ onto $T^\perp$ to determine $(S_a, -\pi_a, \Sigma_a)$ therefore yields a quantity dependent upon $u$. Most importantly,
the reactive component \((S_r, \pi_r, \Sigma_r)\) of \((S, \pi, \Sigma)\) as determined for a particular choice of \(u\) need not be powerless for another choice of \(u\). No explicit expressions for the active and reactive orientational body forces and orientational stress tensors are given by Anderson et al. (1999). To complete the derivation of the constitutive functions for nematic elastomers, we follow the ideas developed by Anderson et al. (1999). Our main point of departure is that we do not use an arbitrary vector to convert the constraint equation (1.3) into a scalar equation, but we retain the vector constraint equation (1.3) when constructing the tangent space \(T\). Our approach amounts to considering the union, over all possible \(u\), of all three-dimensional spaces \(T\) of the kind defined by Anderson et al. (1999). This union is a five-dimensional linear space. The active component of \((S, \pi, \Sigma)\) must be orthogonal to this space. As a consequence of this requirement, we find that

\[
S_a = \left( \mathbb{I} - \frac{1}{|F|^{-2}} F^\top \otimes F^\top \right) S,
\]

\[
\pi_a = (I - n \otimes n) \pi + G(I + G^\top G)^{-1}(\Sigma^\top n - G^\top \pi),
\]

\[
\Sigma_a = \Sigma - n \otimes (I + G^\top G)^{-1}(\Sigma^\top n - G^\top \pi);
\]

further, as potentially useful alternatives to the last two relations, we obtain

\[
\pi_a = (I + GG^\top)^{-1}\{(I - n \otimes n) \pi + G\Sigma^\top n\},
\]

\[
\Sigma_a = (I - n \otimes n) \Sigma + (n \otimes \pi_a) G.
\]

The foregoing results are of an entirely geometrical character and therefore hold regardless of whether there exists an underlying constitutive response function \(\hat{\psi}\) determining the free energy \(\psi\), per unit reference volume, as a function of the deformation gradient \(F = \text{Grad } y\), the orientation \(n\), and the orientation gradient \(G = \text{Grad } n\). Granted the existence of such a function, thermodynamic considerations determine \(S_a\), \(\pi_a\), and \(\Sigma_a\) through appropriate derivatives of \(\hat{\psi}\):

\[
(S_a, -\pi_a, \Sigma_a) = \left( \frac{\partial \hat{\psi}(F, n, G)}{\partial F}, \frac{\partial \hat{\psi}(F, n, G)}{\partial n}, \frac{\partial \hat{\psi}(F, n, G)}{\partial G} \right).
\]

Alternatively, granted an suitably defined smooth extension \(\tilde{\psi}\) of \(\hat{\psi}\) to \(\mathcal{W} \setminus \{O, 0, O\}\), with \(\mathcal{W} = \text{Lin} \times \mathcal{V} \times \text{Lin}\), we have

\[
S_a = \left( \mathbb{I} - \frac{1}{|F|^{-2}} F^\top \otimes F^\top \right) \frac{\partial \tilde{\psi}(F, n, G)}{\partial F},
\]

\[
\pi_a = -(I - n \otimes n) \frac{\partial \tilde{\psi}(F, n, G)}{\partial n}
\]

\[
+ G(I + G^\top G)^{-1}\left\{ \left( \frac{\partial \tilde{\psi}(F, n, G)}{\partial G} \right)^\top n + G^\top \frac{\partial \tilde{\psi}(F, n, G)}{\partial n} \right\},
\]

\[
\Sigma_a = \frac{\partial \tilde{\psi}(F, n, G)}{\partial G}
\]

\[
- n \otimes (I + G^\top G)^{-1}\left\{ \left( \frac{\partial \tilde{\psi}(F, n, G)}{\partial G} \right)^\top n + G^\top \frac{\partial \tilde{\psi}(F, n, G)}{\partial n} \right\}.
\]

With the general structure of the constitutive theory for a uniaxial nematic elastomer settled, we are led naturally to considering a variational framework for the description of equilibrium. This leads to the Euler–Lagrange equations

\[
\text{Div} \left( \frac{\partial \psi(F, n, G)}{\partial F} \right) - F^{\top} \text{Grad} p + b = 0, \\
(I - n \otimes n) \left\{ \text{Div} \left( \frac{\partial \psi(F, n, G)}{\partial G} \right) - \frac{\partial \psi(F, n, G)}{\partial n} + c \right\} = 0,
\]

where \( p \) is the reactive pressure necessary to maintain the constraint (1.4) of incompressibility and \( b \) and \( c \) represent external forces. With reference to the work of Anderson et al. (1999), these partial-differential equations may be interpreted, respectively, as expressions of standard force balance and orientational force balance.

To illustrate some features of the theory, we consider a counterpart of a problem studied by Rivlin (1949): the inflation of tube with cross-section a circular annulus. Subsequent to Rivlin’s work, the problem of the inflation of a cylindrical elastic tube has been considered by a number of researchers. Aside from Rivlin’s original contribution, the works of Haughton & Ogden (1979), Chung et al. (1984), Jafari et al. (1984), Carroll (1988), Hill (1993), and Chen & Haughton (2003) are representative. However, this problem has not yet been studied for a nematic elastomer. For nematic elastomers, the anisotropy of the reference configuration plays an important role in the problem. Here, we take the axis \( n \) of the referential conformation to be radial, in which case the reference configuration is inhomogeneous. Furthermore, we take the asphericity of the referential conformation to be radial, in which case the reference configuration is inhomogeneous. Furthermore, we take the asphericity of the referential conformation \( A \) to be prolate; that is, we take \( q > 0 \). For simplicity, we assume that the parameters \( a \) and \( q \) entering the definition (1.2) of the conformation \( A \) obey \( a = a_0 \) and \( q = q_0 \). Thus, \( A \) differs from \( A_0 \) only in its axis and the conformation at a point in the deformed specimen differs from its counterpart in the reference specimen only by a rigid rotation. Employing the semi-inverse approach of Rivlin (1948, 1949), we stipulate forms for the deformation and orientation fields. Specifically, we assume that

\[
\begin{align*}
y(x) &= \sqrt{(\lambda^2 - 1)r_i^2 + r^2e_r + ze_z}, \quad 1 < \lambda < \infty, \\
n(x) &= \cos \varphi e_r + \sin \varphi e_z, \quad 0 \leq \varphi < \pi,
\end{align*}
\]

where \( r \) is the radial coordinate, \( e_r \) and \( e_z \) are the radial and axial elements of the cylindrical-polar basis at the point \( x \) in the reference configuration, \( r_i \) is the inner radius of the tube, and \( \lambda \) and \( \varphi \) are assignable constants. For definiteness, we work with the two-constant expression of the free-energy function in Anderson et al. (1999), generalized appropriately to account for the inhomogeneity of the referential orientation \( n_0 \). As is the case in Rivlin’s (1949) work, the standard force balance (1.9)_1 determines the pressure required to maintain the constraint of incompressibility in terms of the parameters \( \lambda \) and \( \varphi \). On the other hand, the orientational force balance (1.9)_2 constrains the parameter \( \varphi \) to be equal to either 0 or \( \pi/2 \). Thus, by (1.10)_2, \( n \) either remains radial or switches to axial. The theory therefore gives rise to two solution branches. To explore the stability of these branches, we consider the total potential energy of the specimen. We find

that the solution branch with \( n = e_x \) is stable for \( \lambda \) sufficiently close to unity and that the solution branch with \( n = e_z \) is stable for \( \lambda \) sufficiently large. Using the expression for pressure in terms of \( \lambda \) and \( \varphi \), our stability conditions can be expressed in terms of the dimensionless inflation pressure. For each value of that pressure, we find that there are two equilibrium solutions, one of these solutions being stable at low pressures and the other being stable at high pressures. In addition, the existence of an intermediate range of inflation pressures indicates that during loading the specimen may undergo a discontinuous jump from one solution branch to another. Mechanically induced optical switching of this nature has been observed previously during the stretching on nematic elastomeric sheets (Kundler & Finkelmann 1995; Roberts et al. 1997; Talroze et al. 1999; Zubarev et al. 1999). The application considered here suggests the possibility of using nematic elastomers for pressure-sensitive optical switches.

2. Constraint manifold. Kernel and co-kernel of the derivatives of the constraint functions

Following the notation used by Anderson et al. (1999), we use \( W = \text{Lin} \times \mathcal{V} \times \text{Lin} \) to denote the space of the independent variables of the constitutive functions. Obviously, \( W \) is a 21-dimensional linear space. Define the constraint function \( g : W \rightarrow \mathbb{R} \) by

\[
\gamma(F, n, G) = (\det F - 1, |n|^2 - 1, G^T n),
\]

and define the constraint manifold by

\[
\mathcal{M} = \{(F, n, G) \in W : \gamma(F, n, G) = 0\}.
\]

The constraint manifold is 16-dimensional. Anderson et al. (1999) chose the constraint function to map \( W \) to \( \mathbb{R}^3 \) by taking \( u \cdot G^T n \) as the third component in (2.1), with \( u \) being an arbitrary vector. The constraint manifold so defined is 18-dimensional and depends on \( u \).

The derivative of the constraint function \( \gamma \) at \((F, n, G)\) is a linear transformation \( \Delta \) from \( W \) to \( \mathbb{R} \times \mathbb{R} \times \mathcal{V} \), given by

\[
\Delta(H, t, J) = (H^\top \cdot H, 2n \cdot t, J^\top n + G^\top t).
\]

The kernel of \( \Delta \) is defined by

\[
\ker \Delta = \{(H, t, J) \in W : \Delta(H, t, J) = 0\}.
\]

The co-kernel of \( \Delta \), denoted by \( \text{cok} \Delta \), is the orthogonal complement of \( \ker \Delta \) in \( W \). It is readily observed that \( \ker \Delta \) is a 16-dimensional linear space and that \( \text{cok} \Delta \) a five-dimensional linear space. We thus can write

\[
W = \ker \Delta \oplus \text{cok} \Delta.
\]

When the third component of (2.1) is replaced by the requirement that \( u \cdot G^T n = 0 \) for any vector \( u \), the counterpart of \( \text{cok} \Delta \), which Anderson et al. (1999) denote by \( \mathcal{T} \), is a three-dimensional linear space dependent on the value of \( u \). The linear space \( \text{cok} \Delta \) introduced here is then the union, over all possible \( u \), of all three-dimensional spaces \( \mathcal{T} \) of the kind defined by Anderson et al. (1999).
3. Active and reactive components of stresses and orientational body force

Let \( S \) be the deformatonal stress, \( \pi \) the internal orientational body force density, and \( \Sigma \) be the orientational stress. Following Anderson et al. (1999), we consider the decompositions,

\[
S = S_a + S_r, \quad \pi = \pi_a + \pi_r, \quad \Sigma = \Sigma_a + \Sigma_r,
\]

of these fields into active and reactive components, where

\[
(S_a, -\pi_a, \Sigma_a) \in \ker \Delta \quad \text{and} \quad (S_r, -\pi_r, \Sigma_r) \in \cok \Delta. \tag{3.2}
\]

These requirements stem from the following two considerations.

(i) First, if \( (F(t), n(t), G(t)) \) are differentiable functions of time satisfying the constraint equation \( \gamma(F(t), n(t), G(t)) = 0 \), then we have by (2.1) and (2.3) that

\[
\Delta(F, \dot{n}, \dot{G}) = 0; \tag{3.3}
\]

that is, \( (F, \dot{n}, \dot{G}) \) belongs to \( \ker \Delta \). It then follows from (3.2) that

\[
(S_r, -\pi_r, \Sigma_r) \cdot (F, \dot{n}, \dot{G}) = 0. \tag{3.4}
\]

This implies that the reactive component is powerless on any admissible process, as has been observed by Anderson et al. (1999).

(ii) Second, suppose that there exists a function \( \hat{\psi} \) determining the free-energy density \( \psi \) in terms of \( (F, n, G) \). Such a function must then be defined on the constraint manifold \( \mathcal{M} \). The tangent space of \( \mathcal{M} \) is exactly \( \ker \Delta \). This space is also the domain of the derivative of \( \hat{\psi} \). On assuming that \( S_a, \pi_a, \) and \( \Sigma_a \) are also determined constitutively in terms of \( (F, n, G) \), the standard Coleman–Noll argument for derivation of thermocompatible constitutive equations leads to the conclusion that

\[
(S_a, -\pi_a, \Sigma_a) = \left( \frac{\partial \hat{\psi}(F, n, G)}{\partial F}, \frac{\partial \hat{\psi}(F, n, G)}{\partial n}, \frac{\partial \hat{\psi}(F, n, G)}{\partial G} \right). \tag{3.5}
\]

It therefore follows that \( (S_a, -\pi_a, \Sigma_a) \) must belong to \( \ker \Delta \).

An expression for the reactive component of \( (S, -\pi, \Sigma) \) can be obtained by performing the decomposition (3.1) and (3.2). To this end, we first note that five triples \( (F^{-\top}, 0, O), (O, n, O), \) and \( (O, G\mu_i, n \otimes \mu_i), i=1, 2, 3 \), form a basis for \( \cok \Delta \), where \( \{\mu_1, \mu_2, \mu_3\} \) provides an orthonormal basis for \( \mathcal{V} \). Indeed, for any element \( (H, t, J) \) of \( \ker \Delta \), we have, by (2.3) and (2.4),

\[
\begin{align*}
(F^{-\top}, 0, O) \cdot (H, t, J) &= F^{-\top} \cdot H = 0, \\
(O, n, O) \cdot (H, t, J) &= n \cdot t = 0, \\
(O, G\mu_i, n \otimes \mu_i) \cdot (H, t, J) &= t \cdot G\mu_i + n \cdot J\mu_i = 0.
\end{align*}
\]

\[\{\}

Each one of the five triples \((F^{-T}, 0, O), (O, n, O), \) and \((O, G\mu, n \otimes \mu), i=1, 2, 3,\) therefore belongs to \(\text{cok} \, \mathcal{A}\). Moreover, it is obvious that these triples are linearly independent. By (3.2)_2, we may therefore write
\[
(S_r, -\pi_r, \Sigma_r) = -p(F^{-T}, 0, O) - \eta(O, n, O) + (O, G\mu, n \otimes \mu),
\]
where the scalars \(p\) and \(\eta\) and the vector \(\mu\) are constitutively indeterminate. Equivalently,
\[
S_r = -pF^{-T}, \quad \pi_r = \eta n - G\mu, \quad \Sigma_r = n \otimes \mu.
\]

Substituting (3.8) into (3.1) yields
\[
\begin{align*}
S_a &= S + pF^{-T}, \\
\pi_a &= \pi - \eta n + G\mu, \\
\Sigma_a &= \Sigma - n \otimes \mu.
\end{align*}
\]

To determine \(p, \eta,\) and \(\mu,\) we impose the requirement (3.2)_1. Thus, by (2.3), (2.4) and (3.9), we must have
\[
\begin{align*}
F^{-T} \cdot (S + pF^{-T}) &= 0, \\
n \cdot (\pi - \eta n + G\mu) &= 0, \\
(\Sigma - n \otimes \mu)^T n - G^T(\pi - \eta n + G\mu) &= 0.
\end{align*}
\]

Solving equations (3.10) for \(p, \eta,\) and \(\mu\) and using the resulting expressions in (3.9), we find that
\[
\begin{align*}
S_a &= \left( I - \frac{1}{|F^{-1}|^2} F^{-T} \otimes F^{-T} \right) S, \\
\pi_a &= (I - n \otimes n) \pi + G(I + G^T G)^{-1}(\Sigma^T n - G^T \pi), \\
\Sigma_a &= \Sigma - n \otimes (I + G^T G)^{-1}(\Sigma^T n - G^T \pi),
\end{align*}
\]
where \(I\) is the fourth-order identity tensor and \(U \otimes V\) is the fourth-order tensor with the property that \((U \otimes V) W = (V \cdot W) U\) for all second-order tensors \(W\).

Among other things, we emphasize that the active component \(\pi_a\) of the orientational body force density depends on both \(\pi\) and \(\Sigma\), as does the active component \(\Sigma_a\) of the orientational stress \(\Sigma\). This coupling results because \(G\) is the gradient of \(n\) and \(G^T n = 0\). Moreover, on using (3.11)_2,3 to calculate \(G^T \pi_a\) and \(\Sigma_a^T n\), we find that these quantities are constrained by the relation
\[
G^T \pi_a = \Sigma_a^T n.
\]

Direct but important consequences of (3.11)_2,3 and the identities \((I - n \otimes n)^2 = I - n \otimes n, \ G^T n = 0, \) and \((I - n \otimes n) n = 0\) are that
\[
(I - n \otimes n) \pi = (I - n \otimes n) \pi_a - G\mu \quad \text{and} \quad (I - n \otimes n) \Sigma = (I - n \otimes n) \Sigma_a.
\]

Moreover, (3.11)_3 and \(G^T n = 0\) also yield
\[
\Sigma \cdot G = \Sigma_a \cdot G.
\]
Alternatives to $(3.11)_{2,3}$ can be obtained as follows. First, since
\[(I + GG^\top)G(I + G^\top G)^{-1} = G(I + G^\top)(I + G^\top G)^{-1} = G,\]
we observe that
\[G(I + G^\top G)^{-1} = (I + GG^\top)^{-1} G.\] (3.15)
Thus, by $(3.11)_2$ and $G^\top n = 0$, we find that
\[
\pi_a = (I - n \otimes n) \pi + (I + GG^\top)^{-1} G(S^\top n - G^\top \pi) \\
= (I + GG^\top)^{-1}\{(I + GG^\top)(I - n \otimes n) \pi + G\Sigma^\top n - G G^\top \pi\} \\
= (I + GG^\top)^{-1}\{(I - n \otimes n) \pi + G\Sigma^\top n\}.
\] (3.16)
Next, by the decomposition $(3.1)_3$ of $\Sigma$, the relation $(3.8)_3$ for $\Sigma_1$, and $(3.12)$, we observe that
\[\Sigma^\top n = \Sigma_\alpha n + \Sigma_\alpha^\top n = G^\top \pi_a + \mu,\]
and, thus, by $(3.9)_3$, that
\[\Sigma_a = (I - n \otimes n) \Sigma + (n \otimes \pi_a) G.\] (3.17)

The expressions (3.11) determining the active components $S_a$, $\pi_a$, and $\Sigma_a$ in terms of the primitive quantities $S$, $\pi$, and $\Sigma$ stem solely from the geometric decomposition (2.5). In particular, these expressions hold regardless of whether there exists an underlying free-energy density function $\hat{\psi}$. Suppose, now, that there exists a free-energy density function $\hat{\psi}$ defined on the constraint manifold $\mathcal{M}$. The components of the derivative of $\hat{\psi}$ therefore belong to ker $\mathcal{A}$, as indicated in (3.5). A convenient way to compute the derivative of $\hat{\psi}$ is to extend $\hat{\psi}$ smoothly to $\mathcal{W} \setminus \{O, 0, O\}$, take the derivative of the extended function, and (taking advantage of the analysis leading to (3.11)) restrict the derivative to ker $\mathcal{A}$. Although the derivative of the extended function depends on the particular manner in which $\hat{\psi}$ is extended from $\mathcal{M}$ to $\mathcal{W} \setminus \{O, 0, O\}$, the derivative of $\hat{\psi}$ resulting from the restriction operation described above is unique. Let $\tilde{\psi}$ be a smooth extension of $\hat{\psi}$. We then have
\[
\begin{align*}
S_a &= \left(1 - \frac{1}{|F^{-1}|^2} F^{-\top} \otimes F^{-\top}\right) \frac{\partial \tilde{\psi}(F, n, G)}{\partial F}, \\
\pi_a &= -(I - n \otimes n) \frac{\partial \tilde{\psi}(F, n, G)}{\partial n} \\
&\quad + G(I + G^\top G)^{-1}\left\{\left(\frac{\partial \tilde{\psi}(F, n, G)}{\partial G}\right)^\top n + G^\top \frac{\partial \tilde{\psi}(F, n, G)}{\partial n}\right\}, \\
\Sigma_a &= \frac{\partial \tilde{\psi}(F, n, G)}{\partial G} \\
&\quad - n \otimes (I + G^\top G)^{-1}\left\{\left(\frac{\partial \tilde{\psi}(F, n, G)}{\partial G}\right)^\top n + G^\top \frac{\partial \tilde{\psi}(F, n, G)}{\partial n}\right\}.
\end{align*}
\] (3.18)
A comparison of (3.11) and (3.18) should not lead to the erroneous identification of \((S, -\pi, \Sigma)\) with the triple

\[
\left( \frac{\partial \psi(F, n, G)}{\partial F}, \frac{\partial \psi(F, n, G)}{\partial n}, \frac{\partial \psi(F, n, G)}{\partial G} \right);
\]

however, \((S, -\pi, \Sigma)\) and (3.19) do share the same projection onto \(\text{ker} \ A\). An alternative to the approach taken here would be to define \((S, -\pi, \Sigma)\) by adding to (3.19) the triple \((-qF^{-T}, \gamma n - G\lambda, n \otimes \lambda)\) with \(q, \gamma, \lambda\) being viewed as Lagrange multipliers. The difference between such an approach and the geometric approach described here is that the geometric approach determines the multipliers explicitly.

4. Equations of equilibrium

The equations of equilibrium for a nematic elastomer of the kind under consideration are simply the Euler–Lagrange equations arising from an appropriate potential-energy functional. Let \(\mathcal{B}\) denote the region occupied by the body in a reference configuration and let

\[
\mathcal{A} = \{(y, n) \in \mathcal{C}^1(\mathcal{B}; \mathcal{V} \times \mathcal{V}) : \det \text{Grad} \ y = 1, |n|^2 = 1\}
\]

denote the set of admissible deformations and orientations. The potential-energy functional \(\mathcal{E} : \mathcal{A} \rightarrow \mathbb{R}\) is defined by

\[
\mathcal{E}[y, n] = \int_{\mathcal{B}} \{\hat{\psi}(\text{Grad} \ y, n, \text{Grad} \ n) - b \cdot y - c \cdot n\} dv - \int_{\partial \mathcal{B}} t \cdot y da,
\]

where \(\hat{\psi} : \mathcal{M} \rightarrow \mathbb{R}\) is the free-energy density function, \(b : \mathcal{B} \rightarrow \mathcal{V}\) and \(c : \mathcal{B} \rightarrow \mathcal{V}\) are the prescribed external deformational and orientational body forces, and \(t : \partial \mathcal{B} \rightarrow \mathcal{V}\) is the prescribed (deformational) surface traction. In writing (4.2), we assume that the orientational surface traction on \(\partial \mathcal{B}\) is zero.

A stable deformation/orientation pair \((y, n)\) minimizes \(\mathcal{E}\) over \(\mathcal{A}\). Consider a smooth one-parameter family \((\bar{y}(t), \bar{n}(t))\) of deformation/orientation pairs chosen such that \((y, n) = (\bar{y}(0), \bar{n}(0))\) minimizes \(\mathcal{E}\) over \(\mathcal{A}\). It then follows that

\[
\dot{\mathcal{E}}(0) = \left[ \frac{d}{dt} \mathcal{E}[\bar{y}(t), \bar{n}(t)] \right]_{t=0} = 0.
\]

Using (4.2) and the notation \(\dot{y} = \dot{\bar{y}}(0), \ F = \text{Grad} \ \bar{y}(0), \ \dot{F} = \text{Grad} \ \dot{\bar{y}}(0), \ n = \bar{n}(0), \ G = \text{Grad} \ \bar{n}(0), \ \text{and} \ G = \text{Grad} \ \dot{\bar{n}}(0)\), we find that

\[
\dot{\mathcal{E}}(0) = \int_{\mathcal{B}} \left\{ \frac{\partial \psi(F, n, G)}{\partial F} \cdot \dot{F} + \frac{\partial \psi(F, n, G)}{\partial n} \cdot \dot{n} + \frac{\partial \psi(F, n, G)}{\partial G} \cdot \dot{G} - b \cdot \dot{y} - c \cdot n \right\} dv
\]

\[
- \int_{\partial \mathcal{B}} t \cdot \dot{y} da.
\]
Since $\hat{\psi}$ is defined on the constraint manifold $\mathcal{M}$, its derivative is a linear mapping form $\text{ker} \, \mathcal{A}$ to $\mathbb{R}$ and we have

$$
\left\{ \frac{\partial \hat{\psi}(F, n, G)}{\partial F} \cdot F^{-\top} = 0, \quad \frac{\partial \hat{\psi}(F, n, G)}{\partial n} \cdot n = 0, \right. \\
\left. \left( \frac{\partial \hat{\psi}(F, n, G)}{\partial G} \right)^\top n + G^\top \frac{\partial \hat{\psi}(F, n, G)}{\partial n} = 0. \right\}
$$

(4.5)

Moreover, since $(\dot{y}(t), \hat{n}(t))$ belongs to $\mathcal{A}$ for each $t$, we have

$$(\text{Grad} \, \dot{y}) \cdot F^{-\top} = 0, \quad \hat{n} \cdot n = 0, \quad (\text{Grad} \, \hat{n})^\top n + G^\top \hat{n} = 0. \quad (4.6)$$

Bearing in mind (4.5) and (4.6), we can rewrite (4.4) as

$$
\dot{E}(0) = \int_B \left\{ \left( \frac{\partial \hat{\psi}(F, n, G)}{\partial F} - p F^{-\top} \right) \cdot \dot{F} + \left( \frac{\partial \hat{\psi}(F, n, G)}{\partial n} - \eta n + G \mu \right) \cdot \dot{n}
\right. \\
+ \left. \left( \frac{\partial \hat{\psi}(F, n, G)}{\partial G} + n \otimes \mu \right) \cdot \hat{G} - b \cdot \dot{y} - c \cdot \dot{n} \right\} dv - \int_{\partial B} t \cdot \dot{y} \, da,
$$

(4.7)

for arbitrary $p : B \to \mathbb{R}$, $\eta : B \to \mathbb{R}$, and $\mu : B \to \mathcal{V}$. We define the deformational stress tensor $S$, the internal orientational body force density $\pi$, and the orientational stress tensor $\Sigma$ via

$$
S = \frac{\partial \hat{\psi}(F, n, G)}{\partial F} - p F^{-\top},
\quad \pi = -\frac{\partial \hat{\psi}(F, n, G)}{\partial n} + \eta n - G \mu,
\quad \Sigma = \frac{\partial \hat{\psi}(F, n, G)}{\partial G} + n \otimes \mu.
$$

(4.8)

A comparison of (4.8) with (3.9) shows that, in agreement with the preceding analysis,

$$
S_a = \frac{\partial \hat{\psi}(F, n, G)}{\partial F}, \quad \pi_a = -\frac{\partial \hat{\psi}(F, n, G)}{\partial n}, \quad \Sigma_a = \frac{\partial \hat{\psi}(F, n, G)}{\partial G}.
$$

(4.9)

Substituting (4.8) into (4.7) and invoking the divergence theorem, we find that

$$
\dot{E}(0) = \int_B (S \cdot \dot{F} - \pi \cdot \dot{n} + \Sigma \cdot \dot{G} - b \cdot \dot{y} - c \cdot \dot{n}) dv - \int_{\partial B} t \cdot \dot{y} \, da
$$

$$
= -\int_B \{ \text{Div} \, S + b \} \cdot \dot{y} + \{ \text{Div} \, \Sigma + \pi + c \} \cdot \dot{n} \} dv + \int_{\partial B} \{ (S v - t) \cdot \dot{y} + \Sigma v \cdot \dot{n} \} \, da,
$$

(4.10)

with $v$ being the outward unit normal to $\partial B$. It can be shown that the last expression of (4.10) vanishes for all $y$ and $n$ satisfying the constraint if and only if

$$\text{Div} \, S + b = 0 \quad \text{and} \quad \text{Div} \, \Sigma + \pi + c = 0 \quad (4.11)$$

on the interior of $B$ and

$$Sv = t \quad \text{and} \quad \Sigma v = 0 \quad (4.12)$$
on $\partial B$. While the left sides of (4.11)\textsubscript{2} and (4.12)\textsubscript{2} vanish only up to terms proportional to $n$. Any such terms may, however, be safely absorbed by the arbitrariness of $h$ and $m$. Substituting (4.8) into (4.11) and (4.12) yields

$$\text{Div} \left( \frac{\partial \psi(F, n, G)}{\partial F} \right) - F^{-\top} \text{Grad} \, p + b = 0,$$

$$\text{Div} \left( \frac{\partial \psi(F, n, G)}{\partial G} \right) - \frac{\partial \psi(F, n, G)}{\partial n} + (\text{Div} \, \mu + \eta) n + c = 0 \quad (4.13)$$

and

$$\left( \frac{\partial \psi(F, n, G)}{\partial F} - p F^{-\top} \right) v = t, \quad \left( \frac{\partial \psi(F, n, G)}{\partial G} + n \otimes \mu \right) v = 0. \quad (4.14)$$

Since $p$, $\eta$, and $\mu$ appearing in (4.13) are not constitutively determinate, it may be desirable to eliminate them from the equations of equilibrium. The following equations arise from (4.13) by straightforward calculations:

$$\text{Curl} \left\{ F^{\top} \left( \text{Div} \left( \frac{\partial \psi(F, n, G)}{\partial F} \right) + b \right) \right\} = 0,$$

$$(I - n \otimes n) \left\{ \text{Div} \left( \frac{\partial \psi(F, n, G)}{\partial G} \right) - \frac{\partial \psi(F, n, G)}{\partial n} + c \right\} = 0. \quad (4.15)$$

With reference to the work of Anderson et al. (1999), the partial-differential equations (4.15) (or, equivalently, (4.13)) may be interpreted, respectively, as expressions of standard force balance and orientational force balance.

The equations of equilibrium and the boundary conditions presented here are derived based on minimizing the potential-energy functional (4.2) for which the reference configuration $B$ is bounded and for which the deformational surface traction $t$ is a dead-load that does not depend on the deformation. Nevertheless, the end results are valid when $B$ is unbounded and the surface traction is not a dead-load, as is the case in the application considered next.

5. Application: inflation of a cylindrical tube

The problem of the inflation of a cylindrical elastic tube has been considered by a number of researchers. Aside from the pioneering work of Rivlin (1949), the works of Haughton & Ogden (1979), Chung et al. (1984), Jafari et al. (1984), Carroll (1988), Hill (1993), and Chen & Haughton (2003) are representative. We now consider the behaviour of a tubular uniaxial nematic
elastomer subjected to inflation, using the theory developed in the present work.

We let \( f_0, e_1, e_2, e_3 \) be a right-handed orthonormal frame and consider a uniaxial nematic elastomeric body occupying the tubular referential region

\[
\mathcal{B} = \left\{ \mathbf{x} : r_1 \leq \sqrt{x_1^2 + x_2^2} \leq r_0, \quad -\infty < x_3 < \infty \right\},
\]

(5.1)
corresponding to an annular cylinder of inner radius \( r_1 > 0 \), outer radius \( r_0 > r_1 \), and infinite height. Define cylindrical coordinates \( r, \theta, z \) via

\[
r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3,
\]

(5.2)
with \( 0 \leq \theta \leq 2\pi \), and let \( e_r = \cos \theta \, e_1 + \sin \theta \, e_2, \quad e_\theta = -\sin \theta \, e_1 + \cos \theta \, e_2 \), and \( e_z = e_3 \).

We choose the reference configuration so that the referential orientation \( \mathbf{n}_s \) is radial:

\[
\mathbf{n}_s(\mathbf{x}) = e_r, \quad G_s(\mathbf{x}) = \frac{1}{r} e_\theta \otimes e_\theta.
\]

(5.3)
Further, we suppose that the referential asphericity \( q_s \) is positive:

\[
q_s > 0.
\]

(5.4)
The referential conformation \( \mathbf{A}_s \) defined by (1.1) is therefore prolate about the radial direction and hence inhomogeneous.

\( (a) \) Kinematical assumptions

For simplicity, we assume that the parameters \( a \) and \( q \) entering the definition (1.2) of the conformation tensor \( \mathbf{A} \) obey

\[
a = a_s \quad \text{and} \quad q = q_s.
\]

(5.5)
The conformation at a point in the spatial configuration therefore differs from its referential counterpart only by a rigid rotation. This simplifying assumption is also imposed by Verwey et al. (1996) in their analysis of stripe formation in nematic elastomers. Employing the semi-inverse approach of Rivlin (1959), we assume that the deformation \( \mathbf{y} \) and the orientation \( \mathbf{n} \) are given by

\[
\mathbf{y}(\mathbf{x}) = \sqrt{(\lambda^2 - 1) r_1^2 + r^2} \, e_r + z e_z, \quad 1 < \lambda < \infty,
\]

(5.6)
and

\[
\mathbf{n}(\mathbf{x}) = \cos \varphi \, e_r + \sin \varphi \, e_z, \quad 0 \leq \varphi < \pi,
\]

(5.7)
with \( \lambda \) and \( \varphi \) being assignable constants that correspond, respectively, to the circumferential stretch at the inner surface and the angle between the orientation and the plane perpendicular to the axis. For brevity, we introduce

\[
y(r) = \sqrt{(\lambda^2 - 1) r_1^2 + r^2}.
\]

(5.8)
Under (5.6), the inner and outer radii of the specimen are mapped according to

\[
r_1 \mapsto \lambda r_1 > r_1 \quad \text{and} \quad r_0 \mapsto y(r_0) > r_0.
\]

(5.9)
Further, in view of (5.6), an easy calculation shows that

\[ F(x) = \frac{r}{y(r)} e_r \otimes e_r + \frac{y(r)}{r} e_\theta \otimes e_\theta + e_z \otimes e_z; \tag{5.10} \]

thus, consistent with (2.1), \( F \) obeys the constraint (1.4). Similarly, from (5.7),

\[ G(x) = \frac{\cos \varphi}{r} e_\theta \otimes e_\theta. \tag{5.11} \]

(b) Constitutive assumptions

Additionally, we assume that the specimen is characterized by the particular free-energy density

\[ \psi(F, n, G) = \frac{\mu}{2} \left\{ |F|^2 - \frac{q_*}{1 + q_*} |F^\top n|^2 + q_* |Fn_*|^2 - \frac{q_*^2}{1 + q_*} (n_* \cdot F^\top n)^2 - 3 \right\} 
+ \frac{\kappa q_*^2}{2(1 + q_*)} |F^\top G - (n_* \cdot F^\top n)G_*|^2, \tag{5.12} \]

with \( \mu > 0, \kappa > 0, \) and \( G_* = \text{Grad} \ n_* \). The choice (5.12) corresponds to the two-constant expression of Anderson et al. (1999), generalized appropriately to account for the inhomogeneity of the referential orientation \( n_* \). The first term of (5.12) is the neo-classical free-energy density

\[ \frac{\mu}{2} \{ \text{tr}(A^{-1}F A_x F^\top) - \log \det(A^{-1}A_x) - 3 \} \tag{5.13} \]

of Warner et al. (1988) and Bladon et al. (1994) specialized in accord with the assumed uniaxial forms (1.1) and (1.2) for \( A_* \) and \( \bar{A} \) along with (5.5). (See also Warner & Terentjev (2003).) We identify \( \mu \) as the shear modulus. The second term of (5.12) arises from the gradient energy density\(^1\)

\[ \frac{1}{2} (k_1 - k_2 - k_4) (F \cdot G - (n_* \cdot F^\top n) \text{Div} n_*)^2 + \frac{1}{2} (k_2 + k_4) |F^\top G - (n_* \cdot F^\top n)G_*|^2 
- \frac{1}{4} k_4 \{(F^\top n) \times (F^\top G - (n_* \cdot F^\top n)G_* \}^2 + [(n_* \times) \cdot (F^\top G - (n_* \cdot F^\top n)G_*)]^2 
+ \frac{1}{8} (k_3 - k_2 - k_4) [(F^\top G + G^\top F - (n_* \cdot F^\top n)G_* - (n_* \cdot F^\top n)G_*^\top) n_*] \tag{5.14} \]

of Anderson et al. (1999) on taking the splay, twist, bend, and saddle-splay moduli \( k_1, k_2, k_3 \) and \( k_4 \) to obey \( k_1 = k_2 = k_3 = q_*^2 \kappa / (1 + q_*) \) and \( k_4 = 0 \). The free-energy density function \( \psi \) as given in (5.12) is defined on \( \mathcal{M} \). Let \( \psi \) denote the extension of \( \psi \) to \( \mathcal{W} \setminus \{ O, 0, O \} \) chosen so that \( \psi \) and \( \hat{\psi} \) have the same functional

\(^1\)Given an element \( u \) of \( V \), \( u \times \) denotes the skew-symmetric tensor with the property that \( (u \times)v = u \times v \) for all \( v \) in \( V \).
forms. The various partial derivatives of $\tilde{\psi}$ are then given by

$$\frac{\partial \tilde{\psi}(F, n, G)}{\partial F} = \mu \left\{ F(1 + q_s n_s \otimes n_s) - \frac{q_s}{1 + q_s} n \otimes (F^\top n + q_s (n_s \cdot F^\top n)n_s) \right\}$$

$$+ \frac{\kappa q_s^2}{1 + q_s} \left\{ G(F^\top G - (n_s \cdot F^\top n) G_s)^\top \right.$$  

$$- [G_s \cdot (F^\top G - (n_s \cdot F^\top n) G_s)] n \otimes n_s \left\}, \right.$$ 

$$\frac{\partial \tilde{\psi}(F, n, G)}{\partial n} = -\frac{\mu q_s}{1 + q_s} \{ F(F^\top n + q_s (n_s \cdot F^\top n)n_s) \}$$

$$- \frac{\kappa q_s^2}{1 + q_s} \{ G_s \cdot (F^\top G - (n_s \cdot F^\top n) G_s) \} F n_s,$$

$$\frac{\partial \tilde{\psi}(F, n, G)}{\partial G} = \frac{\kappa q_s^2}{1 + q_s} F(F^\top G - (n_s \cdot F^\top n) G_s).$$

(5.15)

\[ (c) \text{ Deformational force balance. Pressure load} \]

Since the form of the deformation is stipulated via (5.6), the deformational force balance (4.13) serves merely to determine the pressure $p$ developed in reaction to the constraint of incompressibility. We assume that the external deformational body force density vanishes, viz.

$$b = 0.$$  

Then substituting (3.9) into (4.11) gives

$$\text{Grad } p = F^\top \text{Div } S_s. \quad (5.16)$$

The tube is inflated by a pressure load on its inner surface while being traction-free on its outer surface. For notational convenience, we denote the pressure load by $\mu P$, with $P$ necessarily dimensionless. The surface traction $t$ is then given by

$$t = \begin{cases} 
\mu P F^{-\top} e_r, & r = r_1, \\
0, & r = r_o.
\end{cases} \quad (5.17)$$

Substituting (5.17) into (4.12) leads to

$$e_r \cdot F^\top S e_r = \begin{cases} 
-\mu P, & r = r_1, \\
0, & r = r_o.
\end{cases} \quad (5.18)$$

\[ ^2 \text{For the assumed form (5.6) of the deformation, an axial force is also developed in the tube. That force can be computed from the stress tensor } S. \]

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It now follows from (3.9)₁, (5.10), (5.16), and (5.18) that

\[
P = \frac{1}{\mu} \int_{r_i}^{r_o} (e_r \cdot \mathbf{F}^\top \mathbf{S} e_r)' \, dr = \frac{1}{\mu} \int_{r_i}^{r_o} e_r' \cdot \{ \mathbf{F}^\top (\mathbf{S}_a - p \mathbf{F}^{\top}) \}' e_r \, dr
\]

\[
= \frac{1}{\mu} \int_{r_i}^{r_o} e_r' \cdot \{ (\mathbf{F}^\top \mathbf{S} a)' e_r - \mathbf{F}^\top \text{Div} \, \mathbf{S} a \} \, dr
\]

\[
= \frac{1}{\mu} \int_{r_i}^{r_o} \left\{ r \left( \frac{1}{y(r)} \right)' e_r \cdot \mathbf{S} a e_r + \frac{1}{y(r)} e_\theta \cdot \mathbf{S} a e_\theta \right\} \, dr,
\]

(5.19)

where a prime is used to denote differentiation with respect to \( r \). Substituting (3.18)₁ and (5.15)₁ into (5.19), making use of (5.3), (5.7), (5.8), (5.10) and (5.11), and introducing the dimensionless variables

\[
\xi = \frac{r_o}{r_i} > 1 \quad \text{and} \quad \epsilon = \frac{k}{\mu r_i^2} > 0,
\]

we find that the dimensionless inflation pressure is given by

\[
P = \int_{r_i}^{r_o} \left\{ \frac{1}{r} - \frac{(1 + q_\ast \sin^2 \varphi)r^3}{y^4(r)} + \frac{\epsilon q_\ast^2 (\lambda^2 - 1) r_i^4 \cos^2 \varphi \left( \frac{r^2 + y^2(r)}{r^3 y^4(r)} \right)}{\mu (1 + q_\ast)} \right\} \, dr
\]

\[
= \log \xi + \frac{1 + q_\ast \sin^2 \varphi}{2} \left\{ \frac{(\xi^2 - 1)(\lambda^2 - 1)}{\lambda^2 (\xi^2 + \lambda^2 - 1)} + \log \frac{\lambda^2}{\xi^2 + \lambda^2 - 1} \right\}
\]

\[
+ \frac{\epsilon q_\ast^2 (\xi^2 - 1) \cos^2 \varphi}{2(1 + q_\ast)} \left\{ \frac{1}{\xi^2} - \frac{1}{\lambda^2 (\xi^2 + \lambda^2 - 1)} \right\};
\]

(5.21)

Aside from \( \xi, q_\ast, \) and \( \epsilon \), the expression (5.21) for \( P \) involves the parameters \( \lambda \) and \( \varphi \) entering the assumed expressions (5.6) and (5.7) for \( y \) and \( n \). We now use the orientational force balance to determine values of \( \varphi \) appropriate to equilibrium.

(d) Orientational force balance

We assume that the external orientational body force density vanishes, viz.

\[ c = 0. \]

Using (5.15)₂ and (5.15)₃ in (3.18)₂ and (3.18)₃, we calculate the active components \( \mathbf{p}_a \) and \( \Sigma_a \) of the internal orientational body force density and the orientational stress. Specializing those expressions in accord with the expressions (5.3), (5.7), (5.10), and (5.11) for \( \mathbf{n}_a, \mathbf{G}_s, \mathbf{n}, \mathbf{F}, \) and \( \mathbf{G} \), we find from (4.15)₂ that

\[
q_\ast \sin 2 \varphi \left\{ \frac{(1 + q_\ast)r^2}{y^2(r)} - \frac{\epsilon q_\ast^6 (\lambda^2 - 1)^2}{r^4 y^4(r)} - 1 \right\} \mathbf{n}^\perp (\mathbf{x}) = 0,
\]

(5.22)

with \( \epsilon \) as defined in (5.20)₂ and \( \mathbf{n}^\perp (\mathbf{x}) = \sin \varphi \, e_r - \cos \varphi \, e_\theta \). Since (5.22) holds for all \( r \) in \( (a, b) \), we must have \( \sin 2 \varphi = 0 \), i.e.

\[
\varphi = 0 \quad \text{or} \quad \varphi = \frac{\pi}{2}.
\]

(5.23)

For \( \varphi = 0 \), (5.7) yields \( \mathbf{n}(\mathbf{x}) = e_r = n_\ast(\mathbf{x}) \). However, for \( \varphi = \pi/2 \), (5.7) yields \( \mathbf{n}(\mathbf{x}) = e_z \).
Each alternative value of $\phi$ in (5.23) corresponds to an equilibrium solution. We thus have two disconnected solution branches. One (or both) of these may be unstable in the sense that it does not provide a minimum of the total potential energy. We now turn to this issue.

(e) Total potential energy

Granted that $y$ and $n$ are of the forms assumed in (5.6) and (5.7), the free energy and the potential of the pressure load are homogeneous along the axis of the specimen. The total potential energy defined by (4.2) can therefore be replaced by an integral over a cross-section of the tube. We define

$$E(\lambda, \varphi) = \frac{2}{\mu r^2} \int_{r_i}^{r_o} \psi(x) r \, dr - \lambda^2,$$

which is proportional to the net dimensionless potential energy per unit length. The second term in (5.24) is the potential energy of the pressure load, scaled by $\mu$. This potential pertains to an experiment in which the internal pressure is controlled. Other types of experiments are possible. For example, the amount of the enclosed gas can be controlled in an experiment or the tube can be inflated by an incompressible fluid of controlled volume. In such cases, the potential of the load will take a different form than that in (5.24), the loading parameter being different as well. Using the expressions (5.3), (5.7), (5.10), and (5.11) for $n$, $G$, $n$, $F$, and $G$ in (5.12) and (5.24) and making use of (5.21), we find that

$$E(\lambda, \varphi) = (\lambda^2 - 1) \left\{ \log \xi + \frac{1 + q_s \sin^2 \varphi}{2} \log \frac{\lambda^2}{\xi^2 + \lambda^2 - 1} \right\} + \frac{q_s^2(\xi^2 - 1) \sin^2 \varphi}{2(1 + q_s)}$$

$$+ \frac{\epsilon q_s \cos^2 \varphi}{2(1 + q_s)} \left\{ \frac{(\xi^2 - 1)(\lambda^2 - 1)}{\xi^2} \right\} + \log \frac{\xi^2 + \lambda^2 - 1}{\xi^2 \lambda^2} - \lambda^2.$$

A necessary condition for $y$ and $n$ as defined by (5.6) and (5.7) to provide a stable equilibrium is that $E$ attain a minimum at $(\lambda, \varphi)$. This leads to the conditions

$$\begin{align*}
\frac{\partial E(\lambda, \varphi)}{\partial \lambda} &= 0, & \frac{\partial E(\lambda, \varphi)}{\partial \varphi} &= 0, \\
\frac{\partial^2 E(\lambda, \varphi)}{\partial \lambda^2} &\geq 0, & \frac{\partial^2 E(\lambda, \varphi)}{\partial \varphi^2} &\geq 0, \\
\frac{\partial^2 E(\lambda, \varphi)}{\partial \lambda^2} \frac{\partial^2 E(\lambda, \varphi)}{\partial \varphi^2} &\geq \left( \frac{\partial^2 E(\lambda, \varphi)}{\partial \lambda \partial \varphi} \right)^2.
\end{align*}$$

By using (5.25), it can be readily shown that, as one might expect, the conditions (5.26)\textsubscript{1,2} lead to equations (5.21) and (5.23). Furthermore, in view of (5.21), the

second-order partial derivatives of $E$ are given by

$$
\frac{\partial^2 E(\lambda, \varphi)}{\partial \lambda^2} = \frac{2(\xi^2-1)}{\lambda^2(\xi^2+\lambda^2-1)^2} \left\{ (1 + q_s \sin^2 \varphi)(\xi^2 a^2 + \xi^2 + \lambda^2 - 1) \right.
\left. + \frac{\epsilon q_s^2 \cos^2 \varphi(\xi^2 + 2\lambda - 1)}{1 + q_s} \right\},
$$

$$
\frac{\partial^2 E(\lambda, \varphi)}{\partial \varphi^2} = \frac{q_s^2 \cos 2\varphi}{1 + q_s} \left\{ \xi^2 - 1 + \frac{(1 + q_s)(\lambda^2 - 1)}{q_s} \log \frac{\lambda^2}{\xi^2 + \lambda^2 - 1}
\right.
\left. - \epsilon \frac{(\xi^2 - 1)(\lambda^2 - 1)}{\xi^2} + \log \frac{\xi^2 + \lambda^2 - 1}{\xi^2 \lambda^2} \right\},
$$

$$
\frac{\partial^2 E(\lambda, \varphi)}{\partial \lambda \partial \varphi} = q_s \lambda \sin 2\varphi \left\{ \frac{(\xi^2 - 1)(\lambda^2 - 1)}{\lambda^2(\xi^2 + \lambda^2 - 1)} + \log \frac{\lambda^2}{\xi^2 + \lambda^2 - 1}
\right.
\left. - \frac{\epsilon q_s(\xi^2 - 1)}{1 + q_s} \left( \frac{1}{\xi^2} - \frac{1}{\lambda^2(\xi^2 + \lambda^2 - 1)} \right) \right\}. \tag{5.27}
$$

Several observations can now be made.

(i) Bearing in mind (5.4), (5.6)₂, and (5.20), the inequality (5.26)₃ holds for all relevant $(\lambda, \varphi)$. An examination of the right side of (5.27)₃ shows that the right side of (5.26)₅ vanishes for both of the equilibrium values 0 and $\pi/2$ of $\varphi$ (and all relevant $\lambda$). Hence, the satisfaction of the stability conditions (5.26)₃–₅ hinges completely on inequality (5.26)₄.

(ii) An examination of the right side of (5.27)₂ reveals that, for each relevant choice of $\lambda$, (5.26)₄ can be positive only for one of the two of the equilibrium values of $\varphi$. In an experiment corresponding to a particular value of $\lambda$, one of the two equilibrium values of $\varphi$ must be unstable and, thus, unobservable.

(iii) Exactly which of the two equilibrium values of $\varphi$ is unstable for a particular choice of $\lambda$ depends on the sign of the quantity

$$
f(\lambda) = \xi^2 - 1 - \frac{(1 + q_s)(\lambda^2 - 1)}{q_s} \log \frac{\xi^2 + \lambda^2 - 1}{\lambda^2}
\right.
\left. - \epsilon \left\{ \frac{(\xi^2 - 1)(\lambda^2 - 1)}{\xi^2} + \log \frac{\xi^2 + \lambda^2 - 1}{\xi^2 \lambda^2} \right\}. \tag{5.28}
$$

Note that $f$ depends smoothly upon $\lambda$ for all $\xi$, $q_s$, and $\epsilon$. Thus, since $f(1) > 0$, the equilibrium solution corresponding to $\varphi = \pi/2$ is unstable for $\lambda$ sufficiently close to unity. Moreover, since $f(\lambda) < 0$ for $\lambda$ sufficiently large, the equilibrium solution corresponding to $\varphi = 0$ is unstable for $\lambda$ sufficiently large.

(iv) For given values of $\xi$, $q_s$, and $\epsilon$, there is precisely one critical value $\lambda_{cr}$ of $\lambda$ for which $f(\lambda_{cr}) = 0$ and such that

$$
f(\lambda) \begin{cases} >0, & \text{if } \lambda < \lambda_{cr}, \\
<0, & \text{if } \lambda > \lambda_{cr}. \end{cases} \tag{5.29}
$$
That is: when \( \lambda < \lambda_{cr} \), the equilibrium solution with \( \phi = \pi/2 \) is unstable, while when \( \lambda > \lambda_{cr} \), the equilibrium solution with \( \phi = 0 \) is unstable.

(v) It follows from (5.21) that, for a given value of \( \lambda \), the dimensionless inflation pressure \( P \) required for the solution with \( \phi = 0 \) is always greater than that for the solution with \( \phi = \pi/2 \).

Based on the above observations, we can draw the following conclusions concerning the existence of equilibrium solutions and their stabilities. For a given value of the dimensionless inflation pressure \( P \), there are exactly two equilibrium solutions of equations (5.21) and (5.23): \( (\lambda, \phi) = (\lambda_1(P), 0) \) and \( (\lambda, \phi) = (\lambda_2(P), \pi/2) \), with \( \lambda_1(P) < \lambda_2(P) \). There are two values \( P_1 = P|_{(\lambda, \phi) = (\lambda_{cr}, \pi/2)} \) and \( P_2 = P|_{(\lambda, \phi) = (\lambda_{cr}, 0)} \), with \( P_1 < P_2 \), of the dimensionless inflation pressure such that for any \( P < P_1 \), the inequality (5.26) is violated at the equilibrium solution \( (\lambda, \phi) = (\lambda_2(P), \pi/2) \), and that for any \( P > P_2 \), the inequality (5.26) is violated at the equilibrium solution \( (\lambda, \phi) = (\lambda_1(P), 0) \). For any \( P \) with \( P_1 < P < P_2 \), the inequality (5.26) is satisfied for both possible equilibrium solutions. These results are illustrated in figure 1 for the particular parameter values \( \xi = 1.1, q_0 = 0.5 \), and \( \epsilon = 10^{-8} \).

Because they are predicated on the specific class of deformation-orientation pairs \((y, n)\) defined by (5.6) and (5.7), the conditions (5.26) are necessary but not sufficient for stability. To derive sufficient conditions for stability, one must use the entire set \( A \) of admissible \((y, n)\) as defined in (4.1). Hence, the above analysis does not conclude the stability of the equilibrium solution for which (5.26) are satisfied. Such a conclusion can, however, be reached when \( P < P_1 \) or \( P > P_2 \) if the existence of a stable equilibrium is either proved or assumed, since there are only two equilibrium solutions for a given pressure load and one of those is unstable. For \( P_1 < P < P_2 \), both equilibrium solutions may be locally stable. This suggests the anticipated phenomenon that, as the dimensionless inflation pressure \( P \) increases, the radius of the tube will undergo a discontinuous increase consistent with a jump from the stable portion of the branch \( \lambda_1(P) \) to the stable portion of the branch \( \lambda_2(P) \).

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Figure 1. Plot of the circumferential stretch \( \lambda \) at the inner radius of the specimen versus the dimensionless pressure \( P \) for \( \xi = 1.1, q_0 = 0.5 \), and \( \epsilon = 10^{-8} \), in which case \( \lambda_{cr} \approx 1.24555 \).

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3 Granted the reasonable values \( \mu = 10^{-5} \) J m\(^{-3} \) and \( \kappa = 10^{-11} \) J m\(^{-1} \) for the moduli \( \mu \) and \( \kappa \), the choices \( \xi = 1.1 \) and \( \epsilon = 10^{-8} \) correspond to a specimen of inner radius \( r_i = 100 \) \( \mu \) and thickness \( r_o - r_i = 10 \) \( \mu \).
Simultaneous with this increase, the orientation $\varphi$ will jump from 0 to $\pi/2$. Similar jumps should also occur on deflation. Further, hysteresis during processes involving cyclic deflation and inflation cannot be ruled out.

6. Summary

A general constitutive framework for nematic elastomers is presented building on the geometrical approach taken by Anderson et al. (1999). Under this approach, the deformational stress, internal orientational body force density, and orientational stress are decomposed naturally into components belonging to the tangent and normal spaces of the constraint manifold. These components admit physical interpretations as the active and reactive parts of these variables in question. Only the active parts can be prescribed via constitutive equations. A variational derivation of the equations of equilibrium is also presented. While the resulting equations are identical to those derived from balance laws, the variational formulation readily lends itself to a stability analysis based on energetic considerations. As an application of the theory, the inflation of a cylindrical tubular specimen under internal pressure is studied. A special feature of the solution to this problem is the existence of two disconnected solution branches. This degeneracy raises naturally the question of stability of these solutions. A restricted stability analysis is then presented by examining the energy for a special class of deformations/orientations. This stability analysis leads to the conclusion that each of the two solution branches is unstable for some part of the loading process, and hence indicates that during loading the specimen may undergo a discontinuous jump from one solution branch to another. Mechanically induced optical switching of this nature has been observed previously during the stretching of nematic elastomeric sheets (Kundler & Finkelmann 1995; Roberts et al. 1997; Talroze et al. 1999; Zubarev et al. 1999). The application considered here suggests the possibility of using nematic elastomers for pressure-sensitive optical switches. Further development of such applications, as well as a complete stability analysis of equilibrium solutions of nematic elastomers, will be the objects of future investigations.

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References


